JACOB'S LADDERS AND THE ASYMPTOTICALLY APPROXIMATE SOLUTIONS OF A NONLINEAR DIOPHANTINE EQUATION

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ABSTRACT. The nonlinear equation which is connected with the main term of the Hardy-Littlewood formula for $\zeta^2(1/2+it)$ is studied. In this direction I obtain the fine results which cannot be reached by published methods of Balasubramanian, Heath-Brown and Ivic in the field of the Hardy-Littlewood integral.

1. FORMULATION OF THE RESULTS

1.1. Let

$$S(T,U)=2\sum_{n<P}\frac{d(n)}{\sqrt{n}}\frac{\sin\left(\frac{U}{2}\ln\frac{P}{n}\right)}{\frac{U}{2}\ln\frac{P}{n}}\cos\left\{\left(2\pi P+\frac{U}{2}\right)\ln\frac{P}{n}-2\pi P-\frac{\pi}{4}\right\}, \tag{1.1}$$

where $P=T/2\pi$ and $d(n)$ is the number of divisors of n. In this paper I consider the nonlinear diophantine equation

$$\tau=\frac{S(T,U)}{\ln T},\ \tau\in[\eta,1-\eta]\bigcup\{1\} \tag{1.2}$$

in two variables T,U with the parameter τ where

$$T\in[T_0,T_0+U_0],\ U\in\left(0,T_0^{1/6-\epsilon/2}\right],\ U_0=T_0^{1/3+2\epsilon}, \tag{1.3}$$

and $0<\eta$ is a sufficiently small fixed number and $0<T_0$ is a sufficiently big fixed number.

Definition. Let for $\bar{\tau}\in[\eta,1-\eta]\cup\{1\}$ there be a sequence $\{T_0(\bar{\tau})\},\ T_0\to\infty$ and the values $\tilde{T}=\tilde{T}(T_0,\bar{\tau}),\ \tilde{U}=\tilde{U}(T_0,\bar{\tau})$ for which

$$\tilde{T}\in[T_0,T_0+1.1U_0],\ \tilde{U}\in\left(0,T_0^{1/6-\epsilon/2}\right], \tag{1.4}$$

$$\bar{\tau}\sim\frac{S(\tilde{T},\tilde{U})}{\ln\tilde{T}},\ T_0\to\infty$$

is fulfilled. Then the pair $[\tilde{T},\tilde{U}]$ is called the *asymptotically approximate solution* (AAS) of the equation (1.2) for $\tau=\bar{\tau}$.

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1.2. The method of parallel and rotating chords (see [4]-[6]) leads to the proof of the following theorems.

Theorem 1. *For $\tau = 1$ there is the continuum AAS of the equation*

$$1 = \frac{S(T,U)}{\ln T}.$$

The structure of the set of these solutions is such as follows: to each sufficiently big T_0 continuum of AAS corresponds.

Theorem 2. *Let γ denote the sequence of the zeroes of $\zeta(1/2+it)$. Then for each $\tau \in [\eta, 1-\eta]$ there is a continuum of the AAS of the equation*

$$\tau = \frac{S(T,U)}{\ln T}.$$

The structure of the set of these solutions is such as follows: to each sufficiently big γ continuum of AAS corresponds.

Remark 1. It is clear that these results cannot be reached by published methods of Balasubramanian, Heath-Brown and Ivic in the field of the Hardy-Littlewood integral.

This paper is a continuation of the series of works [4]-[11].

2. LEMMAS

2.1.

Lemma 1.

$$\begin{aligned} &\int_T^{T+U} \cos\{2\vartheta(t) - t\ln n\}\mathrm{d}t = \\ &= U\frac{\sin\left(\frac{U}{2}\ln\frac{P}{n}\right)}{\frac{U}{2}\ln\frac{P}{n}}\cos\left\{\left(2\pi P + \frac{U}{2}\right)\ln\frac{P}{n} - 2\pi P - \frac{\pi}{4}\right\} + \\ &+\mathcal{O}\left(\frac{U+U^3}{T}\right),\ P = \frac{T}{2\pi}, \end{aligned} \tag{2.1}$$

for $U > 0$, where

$$\vartheta(t) = -\frac{1}{2}t\ln\pi + Im\left\{\ln\Gamma\left(\frac{1}{4}+\frac{1}{2}it\right)\right\}.$$

Proof. Following the formulae (see [7], pp. 221, 329)

$$\begin{aligned} &\vartheta(t) = \frac{1}{2}t\ln\frac{t}{2\pi} - \frac{1}{2}t - \frac{1}{8}\pi + \mathcal{O}\left(\frac{1}{t}\right), \\ &\vartheta'(t) = \frac{1}{2}\ln\frac{t}{2\pi} + \mathcal{O}\left(\frac{1}{t}\right) \end{aligned} \tag{2.2}$$

we have $(t = T + x,\ x \in [0,U])$

$$\begin{aligned} &2\vartheta(T+x) - (T+x)\ln n = \\ &x\ln\frac{P}{n} + 2\pi P\ln\frac{P}{n} - 2\pi P - \frac{\pi}{4} + \mathcal{O}\left(\frac{1+U+U^3}{T}\right). \end{aligned} \tag{2.3}$$

Then we obtain (2.1) by (2.3). □

2.2.

Lemma 2.

$$\frac{1}{U}\int_U^{T+U} Z^2(t)\mathrm{d}t = S(T,U)+\mathcal{O}\left(\frac{1}{T^{1/6}}\right), \tag{2.4}$$

for $U\leq T^{1/6-\epsilon/2}$.

Proof. Let us remind the Hardy-Littlewood formula (see [12], p. 80)

$$Z^2(t)=2\sum_{n\leq\frac{t}{2\pi}}\frac{d(n)}{\sqrt{n}}\cos\{2\vartheta(t)-t\ln n\}+\mathcal{O}(t^{-1/6}) \tag{2.5}$$

with the Motohashi error term (see [3], p. 125). Since

$$\sum_{\frac{T}{2\pi}\leq n\leq\frac{T+U}{2\pi}}\frac{d(n)}{\sqrt{n}}=\mathcal{O}\left(UT^\epsilon\frac{1}{\sqrt{T}}\right)=\mathcal{O}\left(\frac{U}{T^{1/2-\epsilon}}\right),$$

then

$$\begin{aligned} Z^2(t)=2\sum_{n<P}\frac{d(n)}{\sqrt{n}}\cos\{2\vartheta(t)-t\ln n\}+ \\ \mathcal{O}\left(\frac{1}{T^{1/6}}\right)+\mathcal{O}\left(\frac{U}{T^{1/2-\epsilon}}\right), \end{aligned}$$

and

$$\begin{aligned} \int_T^{T+U} Z^2(t)\mathrm{d}t = 2\sum_{n<P}\frac{d(n)}{\sqrt{n}}\int_T^{T+U}\cos\{2\vartheta(t)-t\ln n\}\mathrm{d}t+ \\ +\mathcal{O}\left(\frac{U}{T^{1/6}}\right)+\mathcal{O}\left(\frac{U^2}{T^{1/2-\epsilon}}\right). \end{aligned} \tag{2.6}$$

Since

$$\frac{U+U^3}{T}\sum_{n<P}\frac{d(n)}{\sqrt{n}}=\mathcal{O}\left(\frac{U+U^3}{T}T^\epsilon\sqrt{T}\right)=\mathcal{O}\left(\frac{U+U^3}{T^{1/2-\epsilon}}\right),$$

then we obtain (2.4) by (2.1), (2.6). □

3. Proofs of the Theorems

3.1. **Proof of Theorem 1.** Let us remind that for every sufficiently big T_0 there is a continuum of pairs

$$[\tilde{T},\tilde{U}]:\ \tilde{T}\in[T_0,T_0+U_0],\ \tilde{U}\in\left(0,T^{1/6-\epsilon/2}\right] \tag{3.1}$$

for which the formula

$$\frac{1}{\tilde{U}}\int_{\tilde{T}}^{\tilde{T}+\tilde{U}} Z^2(t)\mathrm{d}t=\ln\tilde{T}\left\{1+\mathcal{O}\left(\frac{\ln\ln\tilde{T}}{\ln\tilde{T}}\right)\right\} \tag{3.2}$$

is true (see [6], Corollary 2, Remark 4). Then we obtain

$$1+\mathcal{O}\left(\frac{\ln\ln\tilde{T}}{\ln\tilde{T}}\right)=\frac{S(\tilde{T},\tilde{U})}{\ln\tilde{T}}+\mathcal{O}\left(\frac{1}{\tilde{T}^{1/6}\ln\tilde{T}}\right),\ T_0\to\infty \tag{3.3}$$

by (2.4), (3.2). Finally, we obtain the assertion by (3.1), (3.3).

3.2. **Proof of Theorem 2.** First of all the formula

$$\int_{\gamma}^{\gamma+U(\gamma,\alpha)} Z^2(t)\mathrm{d}t = \tau U \ln\gamma \left\{1+\mathcal{O}\left(\frac{\ln\ln\gamma}{\ln\gamma}\right)\right\}, \tag{3.4}$$

$$\tau=\tan[\alpha(\gamma,U)]\in[\eta,1-\eta],\ U(\gamma)=\gamma^{1/3+2\epsilon}+\Delta(\gamma)<1.1\gamma^{1/3+2\epsilon}$$

for rotating chord is true ((3.4) is the dual asymptotic formula which corresponds to [5], (4.4) by [6], (1.2)). It is clear that for every fixed direction of the rotating chord (i.e. for the fixed value $\tau\in[\eta,1-\eta]$) a continuum of parallel chords corresponds. From this set we choose a continual subset such that the condition

$$\tilde{U}<\gamma^{1/6-\epsilon/2} \tag{3.5}$$

(compare with [6], Remark 4) is fulfilled. For this continuum set the formula

$$\frac{1}{\tilde{U}}\int_{\tilde{T}}^{\tilde{T}+\tilde{U}} Z^2(t)\mathrm{d}t = \tau\ln\tilde{T}\left\{1+\mathcal{O}\left(\frac{\ln\ln\tilde{T}}{\ln\tilde{T}}\right)\right\} \tag{3.6}$$

is true (see (3.4)), where

$$\gamma<\tilde{T}<\gamma+1.1U_0(\gamma);\ U_0(\gamma)=\gamma^{1/3+2\epsilon}. \tag{3.7}$$

Next, from (3.6) by (2.4) we obtain

$$\tau=\frac{S(\tilde{T},\tilde{U})}{\ln\tilde{T}}+\mathcal{O}\left(\frac{\ln\ln\tilde{T}}{\ln\tilde{T}}\right),\ \gamma\to\infty. \tag{3.8}$$

Finally, we obtain, by (3.5), (3.7), (3.8) the assertion.

4. Discussion on necessity of a new theory for short and microscopic parts of the Hardy-Littlewood integral

Let us remind the results of Balasubramanian, Heath-Brown and Ivic for the Hardy-Littlewood integral

$$\int_0^T Z^2(t)\mathrm{d}t$$

and for the parts of this integral.

4.1. First of all the Balasubramanian formula

$$\int_0^T Z^2(t)\mathrm{d}t = T\ln T+(2c-1-\ln 2\pi)T+R(T),\ R(T)=\mathcal{O}(T^{1/3+\epsilon}) \tag{4.1}$$

is true (see [1]).

Remark 2. The Good's Ω-theorem (see [2]) implies for the Balasubramanian's formula (4.1) that

$$\limsup_{T\to\infty}|R(T)|=+\infty, \tag{4.2}$$

i.e. the error term in (4.1) is unbounded at $T\to\infty$.

For the short interval the Balasubramanian's formula implies

$$\int_T^{T+U_0} Z^2(t)\mathrm{d}t = U_0\ln T+(2c-\ln 2\pi)U_0+\mathcal{O}(T^{1/3+\epsilon}),\ U_0=T^{1/3+2\epsilon}. \tag{4.3}$$

4.2. Furthermore, let us remind the Heath-Brown's estimate (see [3], (7.20), p. 178)

$$\int_{T-G}^{T+G} Z^2(t)\mathrm{d}t = \mathcal{O}\left\{ G\ln T + G\sum_{K}(TK)^{-\frac{1}{4}}\left(|S(K)| + \right.\right. \tag{4.4}$$
$$\left.\left. +K^{-1}\int_0^K |S(x)|\mathrm{d}x\right) e^{-\frac{G^2K}{T}}\right\}$$

(for definition of used symbols see [3], (7.21)-(7.23)), uniformly for $T^\epsilon \leq G \leq T1/2-\epsilon$. And, finally, we add the Ivic' estimate ([3], (7.26))

$$\int_{T-G}^{T+G} Z^2(t)\mathrm{d}t = \mathcal{O}(G\ln^2 T),\ G\geq T^{1/3-\epsilon_0},\ \epsilon_0 = \frac{1}{108}\approx 0.009. \tag{4.5}$$

Remark 3. It is quite evident that the intervals $[T-G,T+G], G\in(0,1)$, for example, cannot be reached in theories leading to the formula (4.3) of Balasubramanian or to the estimates (4.4) and (4.5) of Heath-Brown and Ivic, respectively.

4.3. In this situation I developed the new theory based on geometric properties of the Jacob's ladders. Let us remind the basic formulae of our theory.

Titchmarsh-Kober-Atkinson (TKA) formula (see [12], p. 141)

$$\int_0^\infty Z^2(t)e^{-2\delta t}\mathrm{d}t = \frac{c-\ln(4\pi\delta)}{2\sin\delta} + \sum_{n=0}^{N} c_n\delta^n + \mathcal{O}(\delta^{N+1}) \tag{4.6}$$

remained as an isolated result for the period of 56 years. We have discovered (see [4]) the nonlinear integral equation

$$\int_0^{\mu[x(T)]} Z^2(t)e^{-\frac{2}{x(T)}t}\mathrm{d}t = \int_0^T Z^2(t)\mathrm{d}t, \tag{4.7}$$

in which the essence of the TKA formula is encoded. Namely, we have shown in [4] that the following almost-exact formula for the Hardy-Littlewood integral (after the period of 90 years)

$$\int_0^T Z^2(t)\mathrm{d}t = \frac{\varphi(T)}{2}\ln\frac{\varphi(T)}{2} + (c-\ln 2\pi)\frac{\varphi(T)}{2} + c_0 + \mathcal{O}\left(\frac{\ln T}{T}\right) \tag{4.8}$$

takes place, where $\varphi(T)$ is the Jacob's ladder, i.e. an arbitrary solution to the nonlinear integral equation (4.7).

Remark 4. In the case of our result (4.8) the error term tends to zero as T goes to infinity, namely

$$\lim_{T\to\infty} r(T) = 0,\ r(T) = \mathcal{O}\left(\frac{\ln T}{T}\right),$$

(compare with (4.2)).

4.4. In the papers [4],[5] I obtained the following additive formula

$$\int_T^{T+U} Z^2(t)\mathrm{d}t = U\ln\left(\frac{\varphi(T)}{2}e^{-a}\right)\tan[\alpha(T,U)] + \mathcal{O}\left(\frac{1}{T^{1/3-4\epsilon}}\right) \tag{4.9}$$

that holds true for short parts of the Hardy-Littlewood integral. Next, in the paper [6] I proved the multiplicative asymptotic formula ($\mu[\varphi]=7\varphi\ln\varphi$)

$$(4.10)\quad \int_T^{T+U} Z^2(t)\mathrm{d}t = U\ln T\tan[\alpha(T,U)]\left\{1+\mathcal{O}\left(\frac{\ln\ln T}{\ln T}\right)\right\},\ U\in\left(0,\frac{T}{\ln T}\right]$$

for short and microscopic parts of the Hardy-Littlewood integral.

Remark 5. The formulae (4.7)-(4.10) - and all corollaries from these (see [5], [6]) - cannot e derived within complicated methods of Balasubramanian, Heath-Brown and Ivic. This proves the necessity of a new method - which is based on elementary geometric properties of Jacob's ladders - to study short and microscopic parts of the Hardy-Littlewood integral.

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